

Module

4

Single-phase AC Circuits

Version 2 EE IIT, Kharagpur

Lesson 13

Representation of Sinusoidal Signal by a Phasor and Solution of Current in R-L-C Series Circuits

In the last lesson, two points were described:

1. How a sinusoidal voltage waveform (ac) is generated?
2. How the average and rms values of the periodic voltage or current waveforms, are computed?

Some examples are also described there. In this lesson, the representation of sinusoidal (ac) voltage/current signals by a phasor is first explained. The polar/Cartesian (rectangular) form of phasor, as complex quantity, is described. Lastly, the algebra, involving the phasors (voltage/current), is presented. Different mathematical operations – addition/subtraction and multiplication/division, on two or more phasors, are discussed.

Keywords: Phasor, Sinusoidal signals, phasor algebra

After going through this lesson, the students will be able to answer the following questions;

1. What is meant by the term, ‘phasor’ in respect of a sinusoidal signal?
2. How to represent the sinusoidal voltage or current waveform by phasor?
3. How to write a phasor quantity (complex) in polar/Cartesian (rectangular) form?
4. How to perform the operations, like addition/subtraction and multiplication/division on two or more phasors, to obtain a phasor?

This lesson forms the background of the following lessons in the complete module of single ac circuits, starting with the next lesson on the solution of the current in the steady state, in R-L-C series circuits.

Symbols

i or $i(t)$ Instantaneous value of the current (sinusoidal form)

I Current (rms value)

I_m Maximum value of the current

\bar{I} Phasor representation of the current

ϕ Phase angle, say of the current phasor, with respect to the reference phasor

Same symbols are used for voltage or any other phasor.

Representation of Sinusoidal Signal by a Phasor

A sinusoidal quantity, i.e. current, $i(t) = I_m \sin \omega t$, is taken up as an example. In Fig. 13.1a, the length, OP, along the x-axis, represents the maximum value of the current I_m , on a certain scale. It is being rotated in the anti-clockwise direction at an angular speed, ω , and takes up a position, OA after a time t (or angle, $\theta = \omega t$, with the x-axis). The vertical projection of OA is plotted in the right hand side of the above figure with respect to the angle θ . It will generate a sine wave (Fig. 13.1b), as OA is at an angle, θ with the x-axis, as stated earlier. The vertical projection of OA along y-axis is $OC = AB =$

$i(\theta) = I_m \sin \theta$, which is the instantaneous value of the current at any time t or angle θ . The angle θ is in rad., i.e. $\theta = \omega t$. The angular speed, ω is in rad/s, i.e. $\omega = 2\pi f$, where f is the frequency in Hz or cycles/sec. Thus,

$$i = I_m \sin \theta = I_m \sin \omega t = I_m \sin 2\pi ft$$

So, OP represents the phasor with respect to the above current, i .

The line, OP can be taken as the rms value, $I = I_m / \sqrt{2}$, instead of maximum value, I_m . Then the vertical projection of OA, in magnitude equal to OP, does not represent exactly the instantaneous value of I , but represents it with the scale factor of $1/\sqrt{2} = 0.707$. The reason for this choice of phasor as given above, will be given in another lesson later in this module.

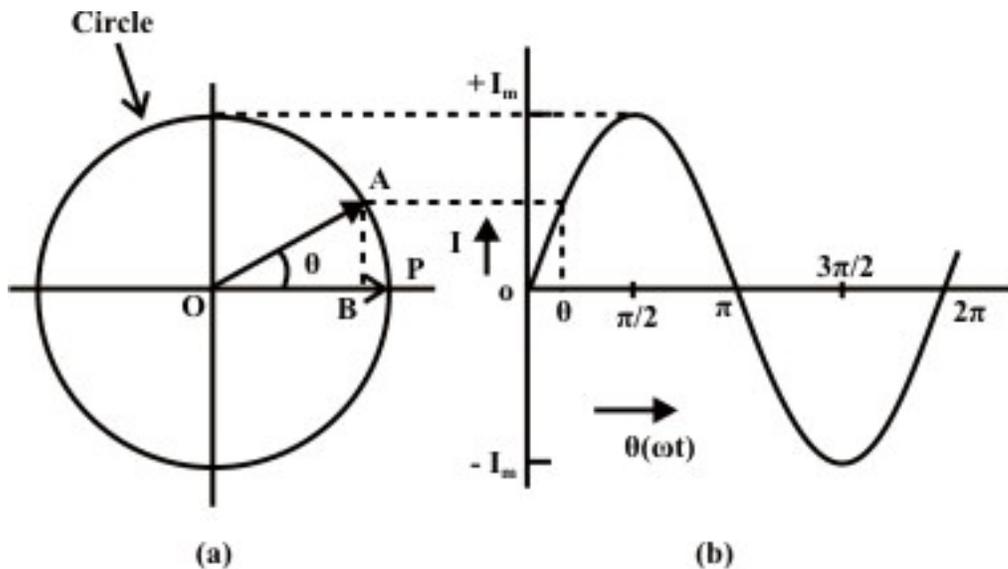


Fig. 13.1(a) Phasor representation of a sinusoidal current, and (b) Waveform

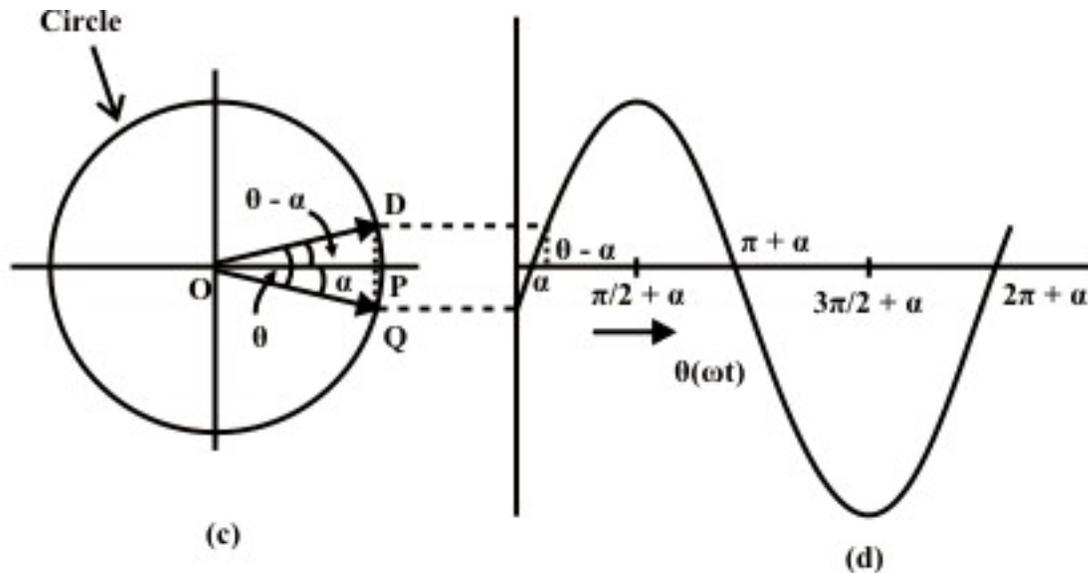


Fig. 13.1 (c) Phasor representation of a phase shifted sinusoidal current, and (d) Waveform

Generalized case

The current can be of the form, $i(t) = I_m \sin(\omega t - \alpha)$ as shown in Fig. 13.1d. The phasor representation of this current is the line, OQ, at an angle, α (may be taken as negative), with the line, OP along x-axis (Fig. 13.1c). One has to move in clockwise direction to go to OQ from OP (reference line), though the phasor, OQ is assumed to move in anti-clockwise direction as given earlier. After a time t , OD will be at an angle θ with OQ, which is at an angle $(\theta - \alpha = \omega t - \alpha)$, with the line, OP along x-axis. The vertical projection of OD along y-axis gives the instantaneous value of the current,

$$i = \sqrt{2} I \sin(\omega t - \alpha) = I_m \sin(\omega t - \alpha).$$

Phasor representation of Voltage and Current

The voltage and current waveforms are given as,

$$v = \sqrt{2} V \sin \theta, \text{ and } i = \sqrt{2} I \sin(\theta + \phi)$$

It can be seen from the waveforms (Fig. 13.2b) of the two sinusoidal quantities – voltage and current, that the voltage, V lags the current I , which means that the positive maximum value of the voltage is reached earlier by an angle, ϕ , as compared to the positive maximum value of the current. In phasor notation as described earlier, the voltage and current are represented by OP and OQ (Fig. 13.2a) respectively, the length of which are proportional to voltage, V and current, I in different scales as applicable to each one. The voltage phasor, OP (V) lags the current phasor, OQ (I) by the angle ϕ , as two phasors rotate in the anticlockwise direction as stated earlier, whereas the angle ϕ is also measured in the anticlockwise direction. In other words, the current phasor (I) leads the voltage phasor (V).

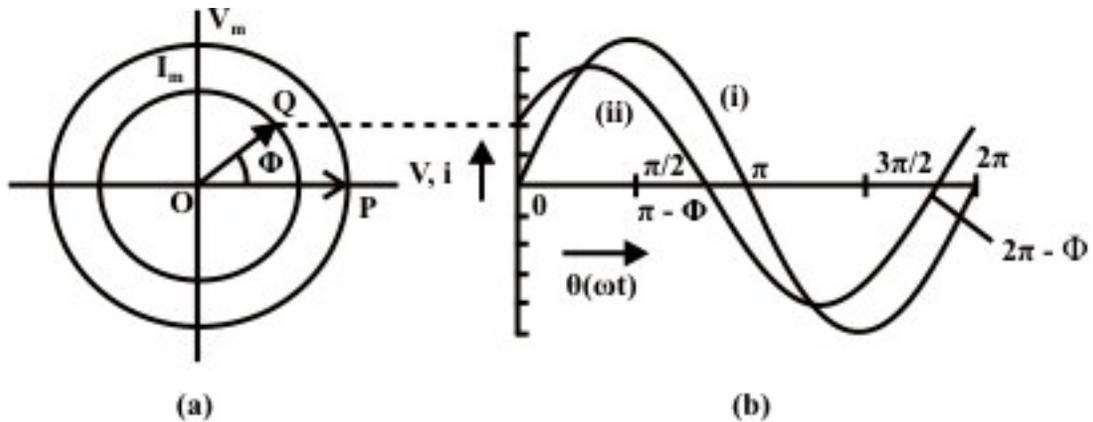


Fig. 13.2 (a) Phasor representation of a sinusoidal (i) voltage and (ii) current, and (b) Waveforms

Mathematically, the two phasors can be represented in polar form, with the voltage phasor (\bar{V}) taken as reference, such as $\bar{V} = V \angle 0^\circ$, and $\bar{I} = I \angle \phi$.

In Cartesian or rectangular form, these are,

$$\bar{V} = V \angle 0^\circ = V + j0, \text{ and } \bar{I} = I \angle \phi = I \cos \phi + jI \sin \phi,$$

where, the symbol, j is given by $j = \sqrt{-1}$.

Of the two terms in each phasor, the first one is termed as real or its component in x-axis, while the second one is imaginary or its component in y-axis, as shown in Fig. 13.3a. The angle, ϕ is in degree or rad.

Phasor Algebra

Before discussing the mathematical operations, like addition/subtraction and multiplication/division, involving phasors and also complex quantities, let us take a look at the two forms – polar and rectangular, by which a phasor or complex quantity is represented. It may be observed here that phasors are also taken as complex, as given above.

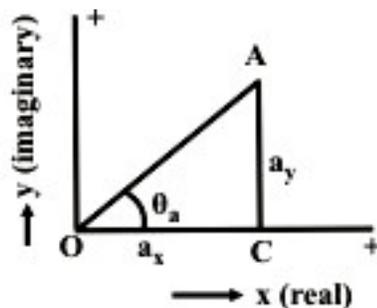


Fig. 13.3 Representation of a phasor, both in rectangular and polar forms

Representation of a phasor and Transformation

A phasor or a complex quantity in rectangular form (Fig. 13.3) is,

$$\bar{A} = a_x + j a_y$$

where a_x and a_y are real and imaginary parts, of the phasor respectively.

In polar form, it is expressed as

$$\bar{A} = A \angle \theta_a = A \cos \theta_a + j A \sin \theta_a$$

where A and θ_a are magnitude and phase angle of the phasor.

From the two equations or expressions, the procedure or rule of transformation from polar to rectangular form is

$$a_x = A \cos \theta_a \text{ and } a_y = A \sin \theta_a$$

From the above, the rule for transformation from rectangular to polar form is

$$A = \sqrt{a_x^2 + a_y^2} \text{ and } \theta_a = \tan^{-1}(a_y / a_x)$$

The examples using numerical values are given at the end of this lesson.

Addition/Subtraction of Phasors

Before describing the rules of addition/subtraction of phasors or complex quantities, everyone should recall the rule of addition/subtraction of scalar quantities, which may be positive or signed (decimal/fraction or fraction with integer). It may be stated that, for the two operations, the quantities must be either phasors, or complex. The example of phasor is voltage/current, and that of complex quantity is impedance/admittance, which will be explained in the next lesson. But one phasor and another complex quantity should not be used for addition/subtraction operation.

For the operations, the two phasors or complex quantities must be expressed in rectangular form as

$$\bar{A} = a_x + j a_y ; \quad \bar{B} = b_x + j b_y$$

If they are in polar form as

$$\bar{A} = A \angle \theta_a ; \quad \bar{B} = B \angle \theta_b$$

In this case, two phasors are to be transformed to rectangular form by the procedure or rule given earlier.

The rule of addition/subtraction operation is that both the real and imaginary parts have to be separately treated as

$$\bar{C} = \bar{A} \pm \bar{B} = (a_x \pm b_x) + j(a_y \pm b_y) = c_x + j c_y$$

$$\text{where } c_x = (a_x \pm b_x) ; \quad c_y = (a_y \pm b_y)$$

Say, for addition, real parts must be added, so also for imaginary parts. Same rule follows for subtraction. After the result is obtained in rectangular form, it can be transformed to polar one. It may be observed that the six values of a 's, b 's and c 's – parts of the two phasors and the resultant one, are all signed scalar quantities, though in the example, a 's and b 's are taken as positive, resulting in positive values of c 's. Also the phase angle θ 's may lie in any of the four quadrants, though here the angles are in the first quadrant only.

This rule for addition can be extended to three or more quantities, as will be illustrated through example, which is given at the end of this lesson.

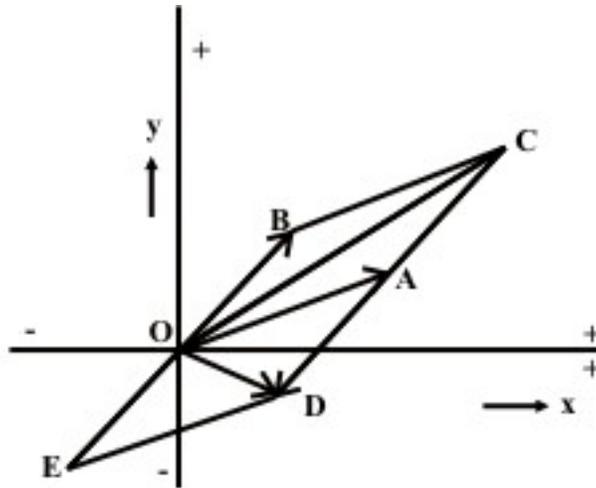


Fig. 13.4 Addition and subtraction of two phasors, both represented in polar form

The addition/subtraction operations can also be performed using the quantities as phasors in polar form (Fig. 13.4). The two phasors are $\bar{A}(OA)$ and $\bar{B}(OB)$. To find the sum $\bar{C}(OC)$, a line AC is drawn equal and parallel to OB . The line BC is equal and parallel to OA . Thus, $\bar{C} = OC = OA + AC = OA + OB = \bar{A} + \bar{B}$. Also,

$$OC = OB + BC = OB + OA$$

To obtain the difference $\bar{D}(OD)$, a line AD is drawn equal and parallel to OB , but in opposite direction to AC or OB . A line OE is also drawn equal to OB , but in opposite direction to OB . Both AD and OE represent the phasor $(-\bar{B})$. The line ED is equal to OA . Thus, $\bar{D} = OD = OA + AD = OA - OB = \bar{A} - \bar{B}$. Also $OD = OE + ED = -OB + OA$. The examples using numerical values are given at the end of this lesson.

Multiplication/Division of Phasors

Firstly, the procedure for multiplication is taken up. In this case no reference is being made to the rule involving scalar quantities, as everyone is familiar with them. Assuming that the two phasors are available in polar form as $\bar{A} = A \angle \theta_a$ and $\bar{B} = B \angle \theta_b$. Otherwise, they are to be transformed from rectangular to polar form. This is also valid for the procedure of division. Please note that a phasor is to be multiplied by a complex quantity only, to obtain the resultant phasor. A phasor is not normally multiplied by another phasor, except in special case. Same is for division. A phasor is to be divided by a complex quantity only, to obtain the resultant phasor. A phasor is not normally divided by another phasor.

To find the magnitude of the product \bar{C} , the two magnitudes of the phasors are to be multiplied, whereas for phase angle, the phase angles are to be added. Thus,

$$\bar{C} = C \angle \theta_c = \bar{A} \cdot \bar{B} = A \angle \theta_A \cdot B \angle \theta_B = (A \cdot B) \angle (\theta_a + \theta_b)$$

where $C = A \cdot B$ and $\theta_c = \theta_a + \theta_b$

Please note that the same symbol, \bar{C} is used for the product in this case.

To divide \bar{A} by \bar{B} to obtain the result \bar{D} , the magnitude is obtained by division of the magnitudes, and the phase is difference of the two phase angles. Thus,

$$\bar{D} = D \angle \theta_d = \frac{\bar{A}}{\bar{B}} = \frac{A \angle \theta_a}{B \angle \theta_b} = \left(\frac{A}{B} \right) \angle (\theta_a - \theta_b)$$

where $D = A/B$ and $\theta_d = \theta_a - \theta_b$

If the phasors are expressed in rectangular form as

$$\bar{A} = a_x + j a_y \text{ and } \bar{B} = b_x + j b_y$$

where $A = \sqrt{(a_x^2 + a_y^2)}$; $\theta_a = \tan^{-1}(a_y / a_x)$

The values of \bar{B} are not given as they can be obtained by substituting b 's for a 's .

To find the product,

$$\bar{C} = C \angle \theta_c = \bar{A} \cdot \bar{B} = (a_x + j a_y) \cdot (b_x + j b_y) = (a_x b_x - a_y b_y) + j(a_x b_y + a_y b_x)$$

Please note that $j^2 = -1$.The magnitude and phase angle of the result (phasor) are,

$$C = \left[(a_x b_x - a_y b_y)^2 + (a_x b_y + a_y b_x)^2 \right]^{1/2} = \sqrt{(a_x^2 + a_y^2)} \cdot \sqrt{(b_x^2 + b_y^2)} = A \cdot B , \text{ and}$$

$$\theta_c = \tan^{-1} \left(\frac{a_x b_y + a_y b_x}{a_x b_x - a_y b_y} \right)$$

The phase angle,

$$\theta_c = \theta_a + \theta_b = \tan^{-1} \left(\frac{a_y}{a_x} \right) + \tan^{-1} \left(\frac{b_y}{b_x} \right) = \tan^{-1} \left[\frac{(a_y / a_x) + (b_y / b_x)}{1 - (a_y / a_x) \cdot (b_y / b_x)} \right]$$

$$= \tan^{-1} \left(\frac{a_x b_y + a_y b_x}{a_x b_x - a_y b_y} \right)$$

The above results are obtained by simplification.

To divide \bar{A} by \bar{B} to obtain \bar{D} as

$$\bar{D} = d_x + j d_y = \frac{\bar{A}}{\bar{B}} = \frac{a_x + j a_y}{b_x + j b_y}$$

To simplify \bar{D} , i.e. to obtain real and imaginary parts, both numerator and denominator, are to be multiplied by the complex conjugate of \bar{B} , so as to convert the denominator into real value only. The complex conjugate of \bar{B} is

$$B^* = b_x + jb_y = B \angle -\theta_b$$

In the complex conjugate, the sign of the imaginary part is negative, and also the phase angle is negative.

$$\bar{D} = d_x + jd_y = \frac{(a_x + ja_y) \cdot (b_x - jb_y)}{(b_x + jb_y) \cdot (b_x - jb_y)} = \left(\frac{a_x b_x + a_y b_y}{b_x^2 + b_y^2} \right) + j \left(\frac{a_y b_x - a_x b_y}{b_x^2 + b_y^2} \right)$$

The magnitude and phase angle of the result (phasor) are,

$$D = \frac{\left[(a_x b_x + a_y b_y)^2 + (a_y b_x - a_x b_y)^2 \right]^{1/2}}{(b_x^2 + b_y^2)} = \frac{\sqrt{(a_x^2 + a_y^2)}}{\sqrt{(b_x^2 + b_y^2)}} = \frac{A}{B}, \text{ and}$$

$$\theta_d = \tan^{-1} \left(\frac{a_y b_x - a_x b_y}{a_x b_x + a_y b_y} \right)$$

The phase angle,

$$\theta_d = \theta_a - \theta_b = \tan^{-1} \left(\frac{a_y}{a_x} \right) - \tan^{-1} \left(\frac{b_y}{b_x} \right) = \tan^{-1} \left(\frac{a_y b_x - a_x b_y}{a_x b_x + a_y b_y} \right)$$

The steps are shown here in brief, as detailed steps have been given earlier.

Example

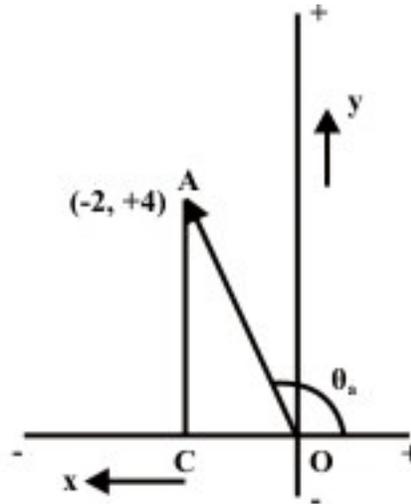


Fig. 13.5 Representation of phasor as an example, both in rectangular and polar forms

The phasor, \bar{A} in the rectangular form (Fig. 13.5) is,

$$\bar{A} = A \angle \theta_a = A \cos \theta_a + j A \sin \theta_a = a_x + ja_y = -2 + j4$$

where the real and imaginary parts are $a_x = -2$; $a_y = 4$

To transform the phasor, \bar{A} into the polar form, the magnitude and phase angle are

$$A = \sqrt{a_x^2 + a_y^2} = \sqrt{(-2)^2 + 4^2} = 4.472$$

$$\theta_a = \tan^{-1}\left(\frac{a_y}{a_x}\right) = \tan^{-1}\left(\frac{4}{-2}\right) = 116.565^\circ = 2.034 \text{ rad}$$

Please note that θ_a is in the second quadrant, as real part is negative and imaginary part is positive.

Transforming the phasor, \bar{A} into rectangular form, the real and imaginary parts are

$$a_x = A \cos \theta_a = 4.472 \cdot \cos 116.565^\circ = -2.0$$

$$a_y = A \sin \theta_a = 4.472 \cdot \sin 116.565^\circ = 4.0$$

Phasor Algebra

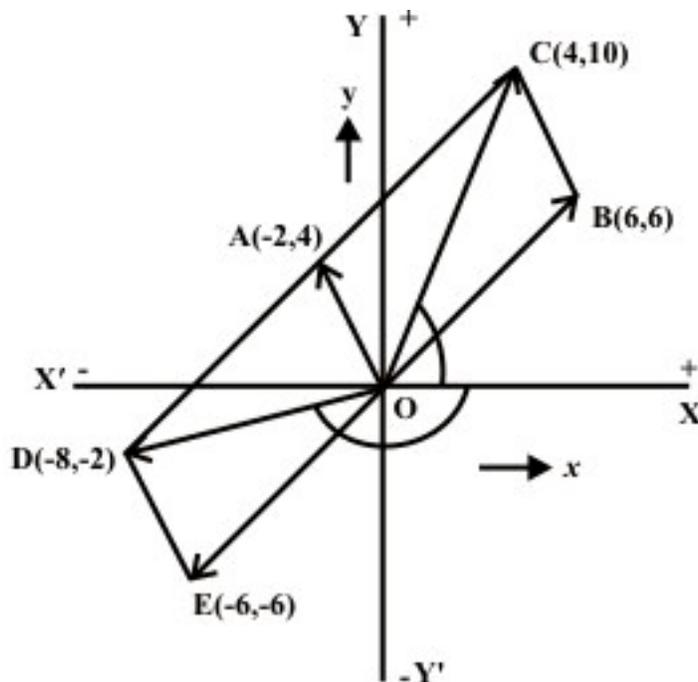


Fig.13.6 Addition and subtraction of two phasors represented in polar form, as an example

Another phasor, \bar{B} in rectangular form is introduced in addition to the earlier one, \bar{A}

$$\bar{B} = 6 + j6 = 8.485 \angle 45^\circ$$

Firstly, let us take the addition and subtraction of the above two phasors. The sum and difference are given by the phasors, \bar{C} and \bar{D} respectively (Fig. 13.6).

$$\bar{C} = \bar{A} + \bar{B} = (-2 + j4) + (6 + j6) = (-2 + 6) + j(4 + 6) = 4 + j10 = 10.77 \angle 68.2^\circ$$

$$\bar{D} = \bar{A} - \bar{B} = (-2 + j4) - (6 + j6) = (-2 - 6) + j(4 - 6) = -8 - j2 = 8.246 \angle -166.0^\circ$$

It may be noted that for the addition and subtraction operations involving phasors, they should be represented in rectangular form as given above. If any one of the phasors

is in polar form, it should be transformed into rectangular form, for calculating the results as shown.

If the two phasors are both in polar form, the phasor diagram (the diagram must be drawn to scale), or the geometrical method can be used as shown in Fig 13.6. The result obtained using the diagram, as shown are the same as obtained earlier.

$$[\bar{C}(\text{OC}) = 10.77, \angle \text{COX} = 68.2^\circ; \text{ and } \bar{D}(\text{OD}) = 8.246, \angle \text{DOX} = 166.0^\circ]$$

Now, the multiplication and division operations are performed, using the above two phasors represented in polar form. If any one of the phasors is in rectangular form, it may be transformed into polar form. Also note that the same symbols for the phasors are used here, as was used earlier. Later, the method of both multiplication and division using rectangular form of the phasor representation will be explained.

The resultant phasor \bar{C} , i.e. the product of the two phasors is

$$\begin{aligned}\bar{C} &= \bar{A} \cdot \bar{B} = 4.472 \angle 116.565^\circ \times 8.485 \angle 45^\circ = (4.472 \times 8.485) \angle (116.565^\circ + 45^\circ) \\ &= 37.945 \angle 161.565^\circ = -36 + j12\end{aligned}$$

The product of the two phasors in rectangular form can be found as

$$\bar{C} = (-2 + j4) \cdot (6 + j6) = (-12 - 24) + j(24 - 12) = -36 + j12$$

The result (\bar{D}) obtained by the division of \bar{A} by \bar{B} is

$$\begin{aligned}\bar{D} &= \frac{\bar{A}}{\bar{B}} = \frac{4.472 \angle 116.565^\circ}{8.485 \angle 45^\circ} = \left(\frac{4.472}{8.485} \right) \angle (116.565^\circ - 45^\circ) = 0.527 \angle 71.565^\circ \\ &= 0.167 + j0.5\end{aligned}$$

The above result can be calculated by the procedure described earlier, using the rectangular form of the two phasors as

$$\begin{aligned}\bar{D} &= \frac{\bar{A}}{\bar{B}} = \frac{-2 + j4}{6 + j6} = \frac{(-2 + j4) \cdot (6 - j6)}{(6 + j6) \cdot (6 - j6)} = \frac{(-12 + 24) + j(24 + 12)}{6^2 + 6^2} \\ &= \frac{12 + j36}{72} = 0.167 + j0.5\end{aligned}$$

The procedure for the elementary operations using two phasors only, in both forms of representation is shown. It can be easily extended, for say, addition/multiplication, using three or more phasors. The simplification procedure with the scalar quantities, using the different elementary operations, which is well known, can be extended to the phasor quantities. This will be used in the study of ac circuits to be discussed in the following lessons.

The background required, i.e. phasor representation of sinusoidal quantities (voltage/current), and algebra – mathematical operations, such as addition/subtraction and multiplication/division of phasors or complex quantities, including transformation of phasor from rectangular to polar form, and vice versa, has been discussed here. The study of ac circuits, starting from series ones, will be described in the next few lessons.

Problems

13.1 Use phasor technique to evaluate the expression and then find the numerical value at $t = 10$ ms.

$$i(t) = 150 \cos(100t - 45^\circ) + 500 \sin(100t) + \frac{d}{dt} [\cos(100t - 30^\circ)]$$

13.2 Find the result in both rectangular and polar forms, for the following, using complex quantities:

a) $\frac{5 - j12}{15 \angle 53.1^\circ}$

b) $(5 - j12) + 15 \angle -53.1^\circ$

c) $\frac{2 \angle 30^\circ - 4 \angle 210^\circ}{5 \angle 450^\circ}$

d) $\left(5 \angle 0^\circ + \frac{1}{3\sqrt{2} \angle -45^\circ} \right) \cdot 2 \angle 210^\circ$

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